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Discussion of the entanglement classification of a 4-qubit pure state

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Abstract. We classify 4-qubit pure states under the stochastic local operation and classical communication (SLOCC). There exist twenty three essentially different classes of states, giving rise to a four-graded partially ordered structure. We also give the criterion to judge which class an arbitrary 4-qubit state belongs to. We re-classify the 4-qubit pure state into $2 \times 2 \times 4, 4 \times 4$ aspects. Finally, we give our analysis of the classification difference of methods for the 3-qubit pure state.

PACS. 03.65.Ud Entanglement and quantum nonlocality (e.g. EPR paradox, Bell's inequalities, GHZ states, etc.) – 03.67.Mn Entanglement production, characterization and manipulation

QICS. 03.05.+c Characterization and classification of entanglement

1 Introduction

The understanding of entanglement is at the heart of quantum information theory (QIT). In view of the central role of entanglement in quantum information processing, it is important to have both a qualitative and quantitative theory of entanglement. Due to the non-local character of the correlations that entanglement induces, it is expected that entanglement is especially important in the context of many body interactions. Despite much effort, however, it has proven exceedingly difficult to gain insight into the structure of multipartite entanglement.

A lot of work has been undertaken to investigate the classification of entanglement of pure multipartite states. To date, a solution has been found to the two-particle pure state entanglement classification under local operations assisted with classical communication (LOCC), which was solved through the local rank [1,2]. Multi-particle entanglement shows much richer structure than the two-particle scenario. Dur et al. gave the classification of the 3-qubit state under stochastic LOCC (SLOCC) [3, 4]. Verstraete et al. exploited group theory to classify the 4-qubit pure state [5]. In particular, Akimasa Miyake gave the onion-like classification of multipartite entangled classes for $2 \times 2 \times 3$ and $2 \times 2 \times 4$ $(2 \times 2 \times n)$ quantum systems based on SLOCC and hyperdeterminants [6–8] using algebraic geometry method. In addition, Karol Zyczkowski and Ingemar Bengtsson introduced quantum entanglement from geometric approach [14] which is very interesting. In this paper, we would like to investigate the classification structure of a 4-qubit pure state and some relevant problems.

The paper is organized as follows: in Section 2 we first review known classification results of the 4-qubit pure state, then we draw some conclusions about the classification structure. We also give a criterion to determine which class an arbitrary 4-qubit state belongs to. In Section 3 we classify the 4-qubit pure state from the $2 \times 2 \times 4$, and 4×4 angles, which also exhibit plentiful and substantial structure. In Section 4 we derive different classification methods of the 3-qubit pure state and give our judgment. Finally, some concluding remarks are summarized in Section 5.

2 Classification structure of the 4-qubit pure state

There are two transformations: local unitary (LU) and SLOCC transformations, which are used to classify states in general. In this paper, we only discuss the SLOCC transformation. SLOCC is a complete positive projective measurement which makes the trace decrease by selecting a series of successful measurement results. In short, the multipartite SLOCC classification is equivalent to the classification of orbits of the natural action: a direct product of special linear groups $SL_{n_1}(C) \times \cdots \times SL_{n_i}(C)$. The theorem in reference [4] states that two multipartite pure states belong to the same class under SLOCC if and only if (iff) they are related by means of an invertible local operator (ILO). The four-qubit pure state can naturally be expressed by SLOCC operations of the form

$$|\Psi'\rangle = A_1 \otimes A_2 \otimes A_3 \otimes A_4 |\Psi\rangle \qquad (1)$$

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Table 1. Entanglement situation of 4-qubit pure state.

State	Four particle entanglement	Three particle entanglement	Two particle entanglement	Separable state entanglement	Hydeterminant $\mathrm{Det}A_4$
G_{abcd}	$a \neq 0, b \neq 0, c \neq 0, d \neq 0$	impossibility	impossibility	impossibility	$\mathrm{Det}A_4 \neq 0$
L_{abc_2}	$a \neq 0, b \neq 0, c \neq 0,$	impossibility	impossibility	$a = b = c = 0$; $\|0000\rangle$	$\mathrm{Det}A_4 = 0$
$L_{a_2b_2}$	$a \neq 0, b \neq 0$	impossibility	$a = b = 0; AC\|EPR\rangle$	impossibility	$\mathrm{Det}A_4 = 0$
L_{ab_3}	yes	impossibility	impossibility	impossibility	$\mathrm{Det}A_4 = 0$
L_{a_4}	yes	impossibility	impossibility	impossibility	$\mathrm{Det}A_4 = 0$
$L_{a_2 o_{3\oplus\bar{1}}}$	$a \neq 0$	$a = 0; \|0\rangle_A \|W\rangle_{BCD}$	impossibility	impossibility	$\mathrm{Det}A_4 = 0$
$L_{o_{5\oplus\bar{3}}}$	yes	impossibility	impossibility	impossibility	$\mathrm{Det}A_4 = 0$
$L_{o_{7\oplus\bar{1}}}$	yes	impossibility	impossibility	impossibility	$\mathrm{Det}A_4 = 0$
$L_{o_{3\oplus\bar{1}} o_{3\oplus\bar{1}}}$	impossibility	$\|0\rangle_A \|GHZ\rangle_{BCD}$	impossibility	impossibility	$\mathrm{Det}A_4 = 0$

with A_i $(i = 1, 2, 3, 4)$ full rank, i.e. invertible 2×2 matrices. By chance, a useful relation can be shown,

$$SL(2, L) \otimes SL(2, L) \simeq SO(4, C) \tag{2}$$

where $SO(4, C)$ denotes the noncompact group of complex orthogonal matrices ($O^T O = I_4$) [10]. reference [4] also shows there is a finite number of classes under SLOCC with Hilbert space $C^2 \otimes C^{n_2} \otimes C^{n_3}$, i.e. having a qubit at least in one of the subsystems. In other cases, there are an infinite number of classes. This has been shown by Frank Verstraete et al. who generalized the singular value decomposition to complex orthogonal matrix equivalent classes, then divided the 4-qubit pure state into nine classes [5,10]

$$\begin{aligned}
G_{abcd} &= \frac{a+b}{2}(|0000\rangle+|1111\rangle) + \frac{a-d}{2}(|0011\rangle+|1100\rangle) \\
&\quad + \frac{b+c}{2}(|0101\rangle+|1010\rangle) + \frac{b-c}{2}(|0110\rangle+|1001\rangle) \\
L_{abc_2} &= \frac{a+b}{2}(|0000\rangle+|1111\rangle\rangle) + \frac{a-b}{2}(|0011\rangle+|1100\rangle) \\
&\quad + c(|0101\rangle + |1010\rangle) + |0110\rangle) = L(1) \\
L_{a_2b_2} &= a(|0000\rangle + |1111\rangle) + b(|0101\rangle + |1100\rangle) \\
&\quad + |0110\rangle + |0011\rangle = L(2) \\
L_{ab_3} &= a(|0000\rangle+|1111\rangle) + \frac{a+b}{2}(|0101\rangle+|1010\rangle) \\
&\quad + \frac{a-b}{2}(|0110\rangle + |1001\rangle) + \frac{i}{\sqrt{2}}(|0001\rangle \\
&\quad + |0010\rangle + |0111\rangle + |1011\rangle) = L(3) \\
L_{a_4} &= a(|0000\rangle + |0101\rangle + |1010\rangle + |1111\rangle) \\
&\quad + (i|0001\rangle + |0110\rangle - i|1011\rangle) = L(4) \\
L_{a_2 o_{3\oplus\bar{1}}} &= a(|0000\rangle + |1111\rangle) + (|0011\rangle \\
&\quad + |0101\rangle + |0110\rangle) = L(5) \\
L_{o_{5\oplus\bar{3}}} &= |0000\rangle + |0101\rangle + |1000\rangle + |1110\rangle = L(6) \\
L_{o_{7\oplus\bar{1}}} &= |0000\rangle + |1011\rangle + |1101\rangle + |1110\rangle = L(7) \\
L_{o_{3\oplus\bar{1}} o_{3\oplus\bar{1}}} &= |0000\rangle + |0111\rangle = L(8).
\end{aligned} \tag{3}$$

Obviously, the fact that the former six classes including parameters indeed elucidates the existence of infinite SLOCC orbits for the 4-qubit pure state.

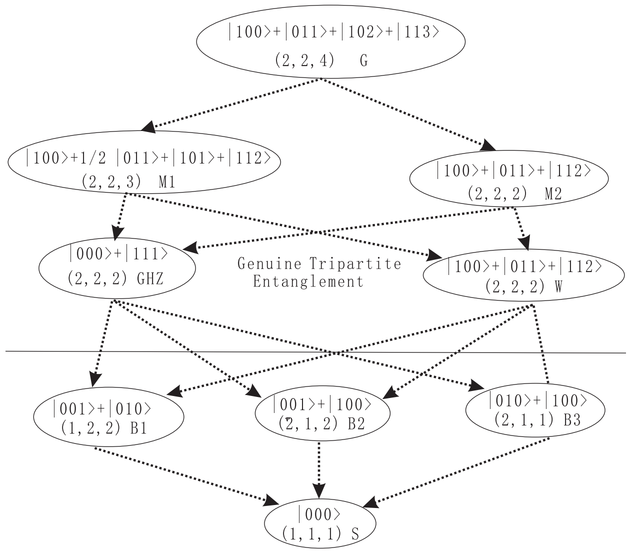

Fig. 1. The partially ordered structure of multipartite pure entangled states in the $2 \times 2 \times n$ $(n \geq 4)$ format, including the 3-qubit case. Each class, corresponding to the SLOCC orbit, is labeled by the representative, local rank, and its name. Non-invertible local operations, indicated by dashed arrows, degrade "higher" entangled classes into "lower" entangled cases.

We note that the representative state for each class of four-qubit states connected by SLOCC operations is in normal form, namely for those pure states, all their reduced local operators are proportional to the identity matrix. By calculating the determinant of reduced density matrices, we explore their entanglement situations. The results are given in Table 1.

As a remark, for six classes with parameters, there are restriction conditions: G_{abcd} with $a \neq 0, b \neq 0, c \neq 0, d \neq 0$; L_{abc_2} with $a \neq 0, b \neq 0, c \neq 0$; $L_{a_2b_2}$ with $a \neq 0, b \neq 0$; $L_{a_2 o_{3\times\bar{1}}}$ with $a \neq 0$; other classes have no restrictions [5].

Besides, references [6–8] give the onion-like classification of SLOCC orbits in the $2 \times 2, k \times k$ $(k \times k', k' \leq k)$, $2 \times 2 \times 2$, $2 \times 2 \times 3$ and $2 \times 2 \times 4$ cases. In order to depict it clearly, the figure of $2 \times 2 \times n$ $(n \leq 4)$ format classification [7,8] is drawn in Figure 1. The aim of this section is to deduce the coarse classification of $2 \times 2 \times 2 \times 2$ quantum systems using the method similar to the 3-qubit case. For

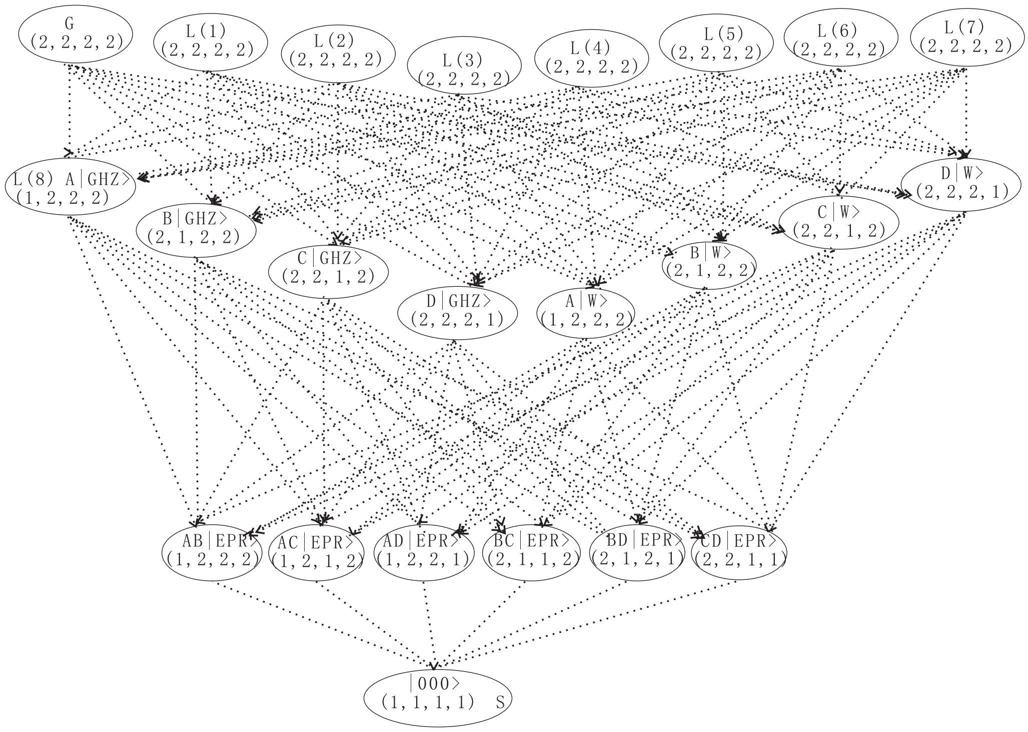

Fig. 2. Twenty three partially ordered structures of 4-qubit pure entangled states. Each class is labeled by the representative and local ranks. SLOCC operations indicated by dashed arrows, degrade "higher" entangled classes into "lower" entangled one.

clarification, the classification structure is depicted concretely in Figure 2.

Now, we summarize the main result by presenting it in the form of a **conclusion**: *consider a pure state in the Hilbert space $C^2 \otimes C^2 \otimes C^2 \otimes C^2$. They are divided into 23 entangled classes, seen in Figure 2, under invertible SLOCC operations. The 23 entangled classes constitute the four-graded partially ordered structure, where non-invertible SLOCC operations degrade higher entangled classes into lower entangled ones. States lying in the same layer are non-invertible, even under probabilistic transformation.*

Then, we may ask how to judge which class an arbitrary 4-qubit state belongs to. Based on the achievements of previous work, we now summary and give a concrete procedure to distinguish these classes. We first present a table to compare some quantities which are different for different classes. Here, we list partial entropy, local rank, and hyperdeteminant, see Table 2.

In Table 2, the $\{i-jkl, i,j,k,l = A,B,C,D\}$ class includes $i|GHZ\rangle_{jkl}, i|W\rangle_{jkl}, (i,j,k,l = A,B,C,D)$. It is worth noting that the classes with a star cannot be classified using only the partial entropy and the hyperdeterminant. But we can settle this problem through the sign of conditional entropy $S(ij|kl), (i,j,k,l = A,B,C,D)$. Specifically, if $S(ij|kl) \geq 0$, this class belongs to the $ij-kl$ class. There exists a special state $|EPR\rangle_{ij} - |EPR\rangle_{kl}$ which has many fascinating properties [7]. Firstly, we give the definition of the mutual entropy

$$S(M:N) = S(M) + S(N) - S(M,N) \qquad (4)$$

Table 2. Values of the local entropies S_A, S_B, S_C, S_D, hyperdeterminant $\mathrm{Det}A_4$ and the local rank.

State	S_A	S_B	S_C	S_D	$\mathrm{Det}A_4$	Local rank
ABCD*	>0	>0	>0	>0	$\neq 0$	$(2,2,2,2)*$
A-BCD	=0	>0	>0	>0	0	$(1,2,2,2)$
B-ACD	>0	=0	>0	>0	0	$(2,1,2,2)$
C-ABD	>0	>0	=0	>0	0	$(2,2,1,2)$
D-ABC	>0	>0	>0	=0	0	$(2,2,2,1)$
AB-CD*	>0	>0	>0	>0	0	$(2,2,2,2)*$
AC-BD*	>0	>0	>0	>0	0	$(2,2,2,2)*$
AD-BC*	>0	>0	>0	>0	0	$(2,2,2,2)*$
A-B-CD	=0	=0	>0	>0	0	$(1,1,2,2)$
A-C-BD	=0	>0	=0	>0	0	$(1,2,1,2)$
A-D-BC	=0	>0	>0	=0	0	$(1,2,2,1)$
B-C-AD	>0	=0	=0	>0	0	$(2,1,1,2)$
B-D-AC	>0	=0	>0	=0	0	$(2,1,2,1)$
C-D-AB	>0	>0	=0	=0	0	$(2,2,1,1)$
A-B-C-D	=0	=0	=0	=0	0	$(1,1,1,1)$

where M, N represent different subsystem, respectively. Then we can judge the type of classification according to the following **Theorem** (Proof in Appendix A).

If $\rho(M,N) = \rho(M) \otimes \rho(N)$ then $S(M:N) = 0$, on the contrary, if $S(M:N) = 0$ then $\rho(M,N) = \rho(M) \otimes \rho(N)$. The steps needed to judge the type of classification for an arbitrary 4-qubit pure state are the following:

Step 1. Calculate the local rank and partial entropy to separate the easiest determined class {A-BCD, B-ACD, C-ABD, D-ABC, A-B-C-D}.

Table 3. Classification structure of a 4-qubit pure state in the $2 \times 2 \times 4$ format.

State/Class		A-B-CD	C-D-AB	A-C-BD	B-D-AC	A-D-BC	B-C-AD
G_{abcd}		G	G	G	G	G	G
L_{abc_2}	$a \neq 0, b \neq 0, c \neq 0$	G	G	G	G	G	G
	$a = b = c = 0$	S	S	S	S	S	S
$L_{a_2b_2}$	$a \neq 0, b \neq 0$	G	G	M_2	M_1	G	G
	$a = b = 0$	B_1	B_1	S	B_3	B_1	B_2
L_{ab_3}		G	G	G	G	G	G
L_{a_4}		G	G	M_2	M_1	G	G
$L_{a_2 o_{3\oplus\bar{1}}}$	$a \neq 0$	M_2	M_1	M_1	M_2	M_1	M_2
	$a = 0$	B_1	W	B_1	W	B_1	W
$L_{o_{5\oplus\bar{3}}}$		M_2	M_2	M_1	M_2	M_1	M_2
$L_{o_{7\oplus\bar{1}}}$		M_2	M_1	M_1	M_1	M_2	M_1
$L_{o_{3\oplus\bar{1}} o_{3\oplus\bar{1}}}$		B_1	GHZ	B_1	GHZ	B_1	GHZ

Step 2. Calculate the conditional entropy or mutual entropy to determine which $ij - kl$ class {AB-CD, AC-BD, AD-BC} the state in the remaining classes belongs to, not including the genuine 4-qubit pure state.

Step 3. For nine inequivalent classes of genuine 4-qubit entanglement, we can use the invariant to judge them. reference [11] gives the invariant $B_{0000}, D^1_{0000}, D^2_{0000}$ and F_{0000} to distinguish six classes including parameters. For the latter three classes without parameters. It is of no use because these invariants are all zero. Nevertheless, the basic covariant knowledge gives us C_{3111}, D_{2200} to distinguish them [12,13].

To sum up, we have presented the classification structure of a 4-qubit pure state, and gave the criterion to distinguish and judge which type of classification for any 4-qubit pure state in theory.

3 Classification of the 4-qubit pure state from $2 \times 2 \times 4$, 4×4 components

In this section, we re-classify the 4-qubit pure state from another perspective. To date, there are many studies for the classification of $C^2 \otimes C^n$ and $C^2 \otimes C^2 \otimes C^n$. They are covered in a theorem [6–8].

Theorem: Consider pure states in the Hilbert space $H = C^2 \otimes C^2 \otimes C^n$. They are divided into nine entangled classes $(G, M_1, M_2, GHZ, W, B_1, B_2, B_3, S)$ under invertible SLOCC operations. These nine entangled classes constitute the five-graded partially ordered structure, where non-invertible SLOCC operations degrade higher entangled classes into lower entangled ones. See Figure 1.

By use of this theorem, we can obtain the concrete classification structure of a 4-qubit pure state in the $2 \times 2 \times 4$ format. Table 3 illustrates that the structure of a re-classified state is plentiful.

If we re-classify the 4-qubit pure state according to $C^n \otimes C^n$, we acquire new results which are listed in Table 4. The state set with local rank $r \leq j (j = 1, ..., k+1)$ is a closed subvariety under SLOCC, the notation S_{j-1} is a singular set of S_j. The outermost general set is $S_{k+1} - S_k$, the inner set is $S_1 = S_1 - S_0$. The result becomes simpler comparing with Table 4.

Table 4. Classification structure of a 4-qubit pure state in the 4×4 format.

State/Class	AB-CD	AC-BD	AD-BC
G_{abcd}	$S_4 - S_3$	$S_4 - S_3$	$S_4 - S_3$
L_{abc_2}	$S_4 - S_3$	$S_4 - S_3$	$S_4 - S_3$
$L_{a_2b_2}$	$S_4 - S_3$	$S_3 - S_2$	$S_4 - S_3$
L_{ab_3}	$S_4 - S_3$	$S_4 - S_3$	$S_4 - S_3$
L_{a_4}	$S_4 - S_3$	$S_3 - S_2$	$S_4 - S_3$
$L_{a_2 o_{3\oplus\bar{1}}}$	$S_3 - S_2$	$S_3 - S_2$	$S_3 - S_2$
$L_{o_{5\oplus\bar{3}}}$	$S_3 - S_2$	$S_3 - S_2$	$S_3 - S_2$
$o_{7\oplus\bar{1}}$	$S_3 - S_2$	$S_3 - S_2$	$S_3 - S_2$
$L_{o_{3\oplus\bar{1}} o_{3\oplus\bar{1}}}$	$S_2 - S_1$	$S_2 - S_1$	$S_2 - S_1$

4 Discussion of different 3-qubit entanglement classifications

Although the classification for the triqubit pure state is quite mature, there always exists some different bifurcations. In general, triqubit pure states are divided into two inequivalent classes: the GHZ-class and the W-class. In particular, the GHZ state $|GHZ\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$ is also called the 3-qubit maximally entangled state, which is widely studied and used in QIT [14].

The quantification of entanglement is well understood for bipartite pure states, however, in a more complex scenario (multipartite systems or mixed states) a complete theory on the quantification of entanglement is still lacking. To date, the quantification of multipartite systems has made great progress. Methods to quantify multipartite entanglement have used either an operational approach [15] or an axiomatic approach [16–18]. These allow determination of the relative entropy of entanglement (RE) [16,17], reversed RE [19], negativity [20,21], reshuffling negativity [22], the robustness of entanglement [18,23], and particularly, the geometric measure of entanglement (GME) [24–28] for which we have given a generalisation [23]. While the entanglement cost [27, 30], the distillable entanglement [31] and the singlet fraction [32,33] belong to operational measures. In addition, there are other measures describing multipartite entanglement in the literature, such as squashed entanglement [34], Hilbert-Schmidt distance [35], n-tangle [36] etc. Without loss of generality, for a relatively simple triplet scenario, the best entanglement quantifier to measure the genuine three particle entanglement is 3-tangle [37,38]

$$\tau = 4|\mathrm{Det}_{222}\Psi| \tag{5}$$

where $\mathrm{Det}_{222}\Psi$ is a hyperdeterminant.

The general expression of the triqubit pure state is

$$|\Psi\rangle = \sum_{i,j,k} t_{i,j,k}|ijk\rangle \qquad i,j,k = 0,1 \tag{6}$$

then the hyperdeterminant is

$$\begin{aligned}\mathrm{Det}_{222}\Psi &= \mathrm{Det}A_3\\ &= t_{000}^2 t_{111}^2 + t_{001}^2 t_{100}^2 + t_{010}^2 t_{101}^2\\ &\quad +4(t_{000}t_{011}t_{101}t_{110} + t_{001}t_{010}t_{100} + t_{111})\\ &\quad -(t_{000}t_{001}t_{110}t_{111} + t_{000}t_{010}t_{101}t_{111}\\ &\quad +t_{000}t_{100}t_{011}t_{111} + t_{001}t_{010}t_{101}t_{110}\\ &\quad +t_{010}t_{100}t_{011}t_{101}).\end{aligned} \tag{7}$$

The value range of 3-tangle is [0, 1] (see Appendix B). It is essential to distinguish different triqubit entangled classes according to 3-tangle. There are two genuine triqubit classes: the GHZ-class and the W-class [4]. When 3-tangle is zero, the state must belong to the W-class; otherwise the state is in the GHZ-class category.

We present the necessary condition of being in the maximally entangled GHZ state for a triqubit pure state form with different parameters. For the general expression of the 3-qubit pure state formula (5), the state is a maximally entangled triqubit state if and only if (iff) 3-tangle $\tau = 1$, and the LU polynomial invariants $I_1 = I_2 = I_3 = 1/2$, $I_4 = 1/4$ [37]. Here

$$\begin{aligned}I_1 = \mathrm{tr}\rho_A^2, \quad I_2 = \mathrm{tr}\rho_c^2, \quad I_3 = \mathrm{tr}\rho_c^2,\\ I_4 = \mathrm{tr}\left((\rho_A \otimes \rho_B)\rho_{AB}\right)\end{aligned} \tag{8}$$

where

$$\begin{aligned}\rho_{AB} = \mathrm{tr}_C|\psi\rangle\langle\psi|, \qquad \rho_A = \mathrm{tr}_{BC}|\psi\rangle\langle\psi|,\\ \rho_B = \mathrm{tr}_{AC}|\psi\rangle\langle\psi|, \quad \rho_C = \mathrm{tr}_{AB}|\psi\rangle\langle\psi|.\end{aligned} \tag{9}$$

In addition, the GHZ-state family of 3-qubit pure states can be expressed as [4]:

$$|\Psi\rangle = \sqrt{K}(\cos\delta|000\rangle + \sin\delta e^{i\varphi}|\varphi_A\varphi_B\varphi_C\rangle) \tag{10}$$

where

$$|\varphi_A\rangle = \cos\alpha|0\rangle + \sin\alpha|1\rangle \tag{11}$$
$$|\varphi_B\rangle = \cos\beta|0\rangle + \sin\beta|1\rangle \tag{12}$$
$$|\varphi_C\rangle = \cos\gamma|0\rangle + \sin\gamma|1\rangle \tag{13}$$

and

$$K = (1 + 2\cos\delta\sin\delta\cos\alpha\cos\beta\cos\gamma\cos\varphi)^{-1} \tag{14}$$

their ranges are

$$K \in \left(\frac{1}{2}, \infty\right), \ \delta \in \left(0, \frac{\pi}{4}\right], \ \alpha, \beta, \gamma \in \left(0, \frac{\pi}{2}\right], \ \varphi \in [0, 2\pi) \tag{15}$$

respectively. It is easy to obtain that the state is in a maximally entangled state if and only if (iff) $\tau = \sin 2\delta(\sin\alpha\sin\beta\sin\gamma - \cos\alpha\cos\beta\cos\gamma\cos\varphi)$, $I_1 = I_2 = I_3 = 1/2$, $I_4 = 1/4$,

As far as the 3-qubit classification is concerned, there still exist different classification methods in the literature [39–41]. Here, we present these differences to elucidate that there are indeed only two kinds of classes for 3-qubit pure state based on the genuine entanglement measure of the triqubit pure state, 3-tangle.

Among these multipartite measures of entanglement, references [39,40] define a genuine multipartite entanglement measure

$$E(\Psi) = \begin{cases} \frac{1}{N}\sum\limits_{i=1}^{N} S_i, & \text{if} \quad S_i \neq 0 \forall i, \\ 0, & \text{if} \quad \text{otherwise,} \end{cases} \tag{16}$$

where

$$S_i = -\mathrm{Tr}[\rho_\psi \log(\rho_\psi)] \tag{17}$$

is the reduced von Neumann entropy. By use of this measure, reference [40] points out the existence of another family except the GHZ-state family and the W-state family for a 3-qubit pure state [4]. They provide an example

$$|\varphi\rangle = x_1|110\rangle + x_2|101\rangle + x_3|011\rangle + x_4|100\rangle \tag{18}$$

when x_1, x_2 are all nonzero, this state is not a GHZ-class state. When x_4 is nonzero, this state is not equivalent to a W-class state. But we calculate its 3-tangle

$$\tau = 4|x_3^2 x_4^2| \tag{19}$$

by selecting appropriate values of x_3, x_4, regardless of the value of x_1, x_2, we can always make the range of 3-tangle belong to $(0, 1]$. Obviously, this state should belong to the GHZ-state family. When x_3 or x_4 is zero, clearly $\tau = 0$, so this state belongs to the W-state family. Besides this state, they also give the state

$$|\phi\rangle = aW_j + be^{i\alpha}W_t + ce^{i\beta}W_i + de^{i\gamma}\widetilde{W}_i \tag{20}$$

where $W, \widetilde{W}(i \neq j \neq t)$ are the basis vectors of a triqubit system.

$$\begin{aligned}&\{W_1 = |000\rangle, W_2 = |110\rangle, W_3 = |101\rangle, W_4 = |011\rangle,\\ &\overline{W}_1 = |111\rangle, \overline{W}_2 = |001\rangle, \overline{W}_3 = |010\rangle, \overline{W}_4 = |100\rangle\}.\end{aligned} \tag{21}$$

The state presented in equation (20) belongs to neither the GHZ-state family nor the W-state family when they choose $|a| = |b| = 0.462157$, $|c| = 0.653614$,

$|d| = 0.381546$ with arbitrary phase. But we compute 3-tangle $\tau = 4|c^2 d^2 e^{2i(\beta+\gamma)}| = 4|c|^2|d|^2 = 0.997535$, which clearly shows this state belongs to the GHZ-class. Also the state

$$|\psi\rangle = a|000\rangle + be^{i\alpha}|110\rangle + ce^{i\beta}|101\rangle + de^{i\gamma}|011\rangle + fe^{i\sigma}\widetilde{W_i} \quad (22)$$

when the subscript of W_i is 1, the 3-tangle is

$$\begin{aligned}\tau &= 4|a^2 f^2 e^{2i\theta} + 4abcde^{i(\alpha+\beta+\gamma)}| \\ &= 4[a^4 f^4 + 16a^2b^2c^2d^2 + 8a^3bcdf^2\cos(2\theta-\alpha-\beta-\gamma)].\end{aligned} \quad (23)$$

Choosing $|a| = 2/3, |b| = |c| = |d| = 1/3, |f| = \sqrt{2}/3$ with arbitrary phase factor, we obtain $\tau = 64/81\cos((2\theta-\alpha-\beta-\gamma)/2) \in [0, 0.79]$. When we choose an appropriate phase angle to make $\cos((2\theta-\alpha-\beta-\gamma)/2) = 0$, this state belongs to the W-class; otherwise this state belongs to the GHZ-class. However reference [17] claims that this state belongs purely to the W type state, in contradiction with our results.

Finally, reference [41] shows a state system expressed as

$$|\psi\rangle = \lambda_1|001\rangle + \lambda_2|010\rangle + \lambda_3|100\rangle + \lambda_4|111\rangle \quad (24)$$

which belongs to neither the GHZ type nor the W type state. Similarly, we can use 3-tangle to distinguish this state. The 3-tangle is

$$\tau = 4|\lambda_1\lambda_2\lambda_3\lambda_4|. \quad (25)$$

When one of the four coefficients $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ is zero, $\tau = 0$, then this state falls into the category of the W class. When $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 1/2$, this state is a maximally entangled state. In other cases the 3-tangle falls into $\tau \in (0, 1)$, this state belongs to the GHZ-state family. All in all, the coefficients of this state determine which class the state belongs to. Hence we think the classification method of the GHZ type and the W type is more reasonable than other classification techniques.

5 Conclusion

The paper discusses the classification structure of 3-qubit and 4-qubit pure states under SLOCC in detail. Nonetheless, this work is useful for developing a comprehensive understanding of the classification structure of three and four particles, and we regard this work as a foundation for exploiting the more general multi-particle cases. We first review some achievable results and deduce our conclusion that there exist twenty three essentially different classes of states, giving rise to a four-graded partially ordered structure for the 4-qubit pure state. We also give the criteria necessary to judge which class an arbitrary 4-qubit state belongs to. In addition, we re-classify the 4-qubit pure state in terms of $2 \times 2 \times 4$, and 4×4 aspects, respectively. Finally, we analyze different classification phenomena of the triqubit pure state and give our interpretation. For mixed state classification, the 3-qubit case is generally solved by Acin et al. [42]. Multipartite mixed state classification is a formidable task, we hope to make some progress in classifying 4-qubit mixed states in future work.

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Appendix A: Proof of theorem

In this appendix, we prove the **Theorem**: Bipartite pure state $|NM\rangle$ with dimension of M and N (the dimension is not restricted), is an entangled state if and only if (iff) the conditional entropy $S(M|N) < 0$.

Proof: According to the property of entropy: $S(N, M) = 0$ for pure state and non-negativity of reduced von Neumann entropy $S(N) > 0$, we obtain the conditional entropy $S(M|N) = S(N, M) - S(N) = -S(N) < 0$. Contrarily, if $S(M|N) < 0$ we have $S(N, M) < S(N)$, by use of the relation $S(N, M) = 0$, we can obtain $S(N) > 0$. Thus we deduce state $|NM\rangle$ is an entangled state because the partial entropy of the separable state is just zero. The proof finishes. □

Appendix B: The range of 3-tangle

In this appendix, we prove the range of 3-tangle is [0, 1] from a new angle. Without loss of generality, the expression of 3-qubit general GHZ-class state is

$$|\Psi\rangle = \sqrt{K}(\cos\delta|000\rangle + \sin\delta e^{i\varphi}|\varphi_A\varphi_B\varphi_C\rangle) \quad (26)$$

where

$$|\varphi_A\rangle = \cos\alpha|0\rangle + \sin\alpha|1\rangle \quad (27)$$
$$|\varphi_B\rangle = \cos\beta|0\rangle + \sin\beta|1\rangle \quad (28)$$
$$|\varphi_C\rangle = \cos\gamma|0\rangle + \sin\gamma|1\rangle \quad (29)$$

and

$$K = (1 + 2\cos\delta\sin\delta\cos\alpha\cos\beta\cos\gamma\cos\varphi)^{-1} \quad (30)$$

their ranges are:

$$K \in \left(\frac{1}{2}, \infty\right), \ \delta \in \left(0, \frac{\pi}{4}\right], \ \alpha, \beta, \gamma \in \left(0, \frac{\pi}{2}\right], \ \varphi \in [0, 2\pi) \quad (31)$$

respectively, then the 3-tangle is expressed as:

$$\tau = \frac{4\cos^2\delta\sin^2\delta\sin^2\alpha\sin^2\beta\sin^2\gamma}{(1 + 2\cos\delta\sin\delta\cos\alpha\cos\beta\cos\gamma\cos\varphi)^2} \quad (32)$$

dependent on the condition

$$\begin{aligned}\cos^2\alpha + \sin^2\alpha = 1 \\ \cos^2\beta + \sin^2\beta = 1 \\ \cos^2\gamma + \sin^2\gamma = 1.\end{aligned} \quad (33)$$

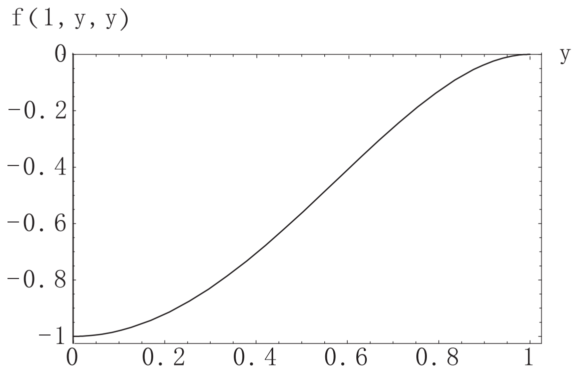

Fig. B.1. Function f is monotonically increasing with the variable y.

By the Language multiplier method, we obtain when $\delta = \pi/4, \varphi = \pi, \tau$ reaches a maximum independent of the values of α, β, γ. That is

$$\tau = \frac{\sin^2\alpha \sin^2\beta \sin^2\gamma}{(1-\cos\alpha\cos\beta\cos\gamma)^2}. \tag{34}$$

Here, we want to prove $\tau = ((\sin^2\alpha\sin^2\beta\sin^2\gamma)/(1-\cos\alpha\cos\beta\cos\gamma)^2) \leq 1$.

Let us call $\cos\alpha = x, \cos\beta = y, \cos\gamma = z$, obviously $0 \leq x, y, z < 1$, thus we have to prove that $((1-x^2)(1-y^2)(1-z^2)/(1-xyz)^2) \leq 1$, i.e. simplify to show that

$$\begin{aligned} &2x^2y^2z^2 - 2xyz - x^2y^2 - x^2z^2 - y^2z^2 + x^2 + y^2 + z^2 \geq 0 \\ &2xyz + x^2y^2 + x^2z^2 + y^2z^2 - x^2 - y^2 - z^2 - 2x^2y^2z^2 \geq 0. \end{aligned} \tag{35}$$

Let $f(x,y,z) = 2xyz + x^2y^2 + x^2z^2 + y^2z^2 - x^2 - y^2 - z^2 - 2x^2y^2z^2$, then we take the derivatives of $f(x,y,z)$ with respect to x, y, z respectively (which we denote by f_x, f_y, f_z and set them to zero)

$$\begin{aligned} f_x &= 2yz + 2xy^2 + 2xz^2 - 2x - 4xy^2z^2 = 0 \\ f_y &= 2xz + 2yx^2 + 2yz^2 - 2y - 4yx^2z^2 = 0 \\ f_z &= 2xy + 2zx^2 + 2zy^2 - 2z - 4zx^2y^2 = 0. \end{aligned} \tag{36}$$

We immediately observe by considering linear combination of the resulting equations $xf_x - yf_y = 0$, that is

$$2(x^2 - y^2)(z^2 - 1) = 0. \tag{37}$$

For a maximum we must have $x = y = z$.

The reason: due to the fact that $f(x,y,z)$ is invariant under permutations of the variables, we only have to check two of the surfaces, e.g. the surfaces specified by $x = 0, x = 1$ (actually $x = 1 - \epsilon$, where ϵ is an infinitesimally small positive number).

(1) $f(0,y,z) = y^2z^2 - y^2 - z^2 \leq 0$.
The maximum in this case is obtained for $y = 0, z = 1-\epsilon$ or $z = 0, y = 1-\epsilon$ (derivative method);

(2) $f(1,y,z) = -(yz-1)^2 \leq 0$.
It can be checked that a necessary condition for a maximum in the second case is $y = z$, see Figure B.1, we see $f(x,y,z) = -(y^2-1)^2$ is monotonically increasing in $y \in [0,1)$, and is thus maximized for $y = z = 1-\epsilon$.

So, we obtain $f(x,y,z) \leq f(1, 1-\epsilon, 1-\epsilon) < 0$. i.e. $\tau_{max} = 1$ as desired. Thus the proof comes to an end. □

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